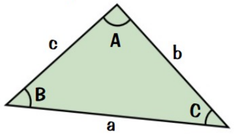
Knowledge Organiser: Year 10 Higher (Summer)

The Sine and Cosine Rules

1

Labelling the Triangle

This is very important. You must label the sides and angles properly so that the letters for the sides and angles correspond with each other. Use lower case letters for the sides and capitals for the angles.



Remember, side 'a' is opposite angle A etc.

It doesn't matter which sides you decide to call a, b, and c, just as long as the angles are then labelled properly.

Three Formulas to Learn: (Grade 7)

The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

You don't use the whole thing with both '=' signs of course, so it's not half as bad as it looks — you just choose the two bits that you want:

e.g. $\frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{a}{\sin A} = \frac{b}{\sin B}$

The Cosine Rule

The 'normal' form is...

$$a^2 = b^2 + c^2 - 2bc \cos A$$

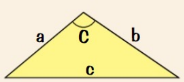
...or this form is good for finding an angle (you get it by rearranging the 'normal' version):

$$\text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Area of the Triangle

This formula comes in handy when you know two sides and the angle between them:

Area of triangle = ½ ab sin C

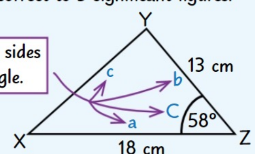


Of course, you already know a simple formula for calculating the area using the base length and height (see p.82). The formula here is for when you don't know those values.

EXAMPLE:

Triangle XYZ has XZ = 18 cm, YZ = 13 cm and angle XZY = 58°. Find the area of the triangle, giving your answer correct to 3 significant figures.

Label the sides and angle.



$\text{Area} = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 18 \times 13 \times \sin 58^\circ$

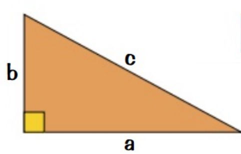
$= 99.2\ cm^2$ (3 s.f.)

Don't forget the units.

SOH: O / S × H

CAH: A / C × H

TOA: O / T × A



$a^2 + b^2 = c^2$

The Sine and Cosine Rules

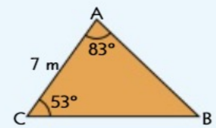
2

There are four main question types where the sine and cosine rules would be applied. So learn the exact details of these four examples and you'll be laughing. WARNING: if you laugh too much people will think you're crazy.

The Four Examples (Grade 7)

1 TWO ANGLES given plus ANY SIDE — SINE RULE needed.

Find the length of AB for the triangle below.



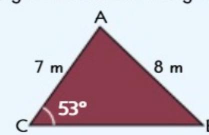
1) Don't forget the obvious... $B = 180° - 83° - 53° = 44°$

2) Put the numbers into the sine rule. $\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{7}{\sin 44^\circ} = \frac{c}{\sin 53^\circ}$

3) Rearrange to find c. $\Rightarrow c = \frac{7 \times \sin 53^\circ}{\sin 44^\circ} = 8.05$ m (3 s.f.)

2 TWO SIDES given plus an ANGLE NOT ENCLOSED by them — SINE RULE needed.

Find angle ABC for the triangle shown below.



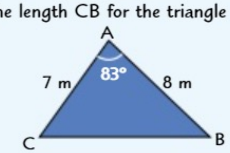
1) Put the numbers into the sine rule. $\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{7}{\sin B} = \frac{8}{\sin 53^\circ}$

2) Rearrange to find sin B. $\Rightarrow \sin B = \frac{7 \times \sin 53^\circ}{8} = 0.6988...$

3) Find the inverse. $\Rightarrow B = \sin^{-1}(0.6988...) = 44.3°$ (1 d.p.)

3 TWO SIDES given plus the ANGLE ENCLOSED by them — COSINE RULE needed.

Find the length CB for the triangle shown below.



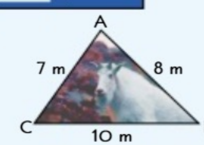
1) Put the numbers into the cosine rule. $a^2 = b^2 + c^2 - 2bc \cos A$
$= 7^2 + 8^2 - 2 \times 7 \times 8 \times \cos 83^\circ$
$= 99.3506...$

2) Take square roots to find a. $a = \sqrt{99.3506...}$
$= 9.97$ m (3 s.f.)

You might come across a triangle that isn't labelled ABC — just relabel it yourself to match the sine and cosine rules.

4 ALL THREE SIDES given but NO ANGLES — COSINE RULE needed.

Find angle CAB for the triangle shown.



1) Use this version of the cosine rule. $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

2) Put in the numbers. $= \frac{49 + 64 - 100}{2 \times 7 \times 8}$

3) Take the inverse to find A. $= \frac{13}{112} = 0.11607...$
$\Rightarrow A = \cos^{-1}(0.11607...)$
$= 83.3°$ (1 d.p.)

Algebraic Fractions

3

Simplifying Algebraic Fractions

Grade 6

You can simplify algebraic fractions by cancelling terms on the top and bottom — just deal with each letter individually and cancel as much as you can. You might have to factorise first (see pages 19 and 25-26).

EXAMPLES:

1. Simplify $\frac{21x^3y^2}{14xy^3}$

÷7 on the top and bottom
÷x on the top and bottom to leave x^2 on the top
÷y^2 on the top and bottom to leave y on the bottom

$$\frac{21x^3y^2}{14xy^3} = \frac{3x^2}{2y}$$

2. Simplify $\frac{x^2 - 16}{x^2 + 2x - 8}$

Factorise the top using D.O.T.S.

Factorise the quadratic on the bottom

$$\frac{(x+4)(x-4)}{(x-2)(x+4)} = \frac{x-4}{x-2}$$

Then cancel the common factor of $(x + 4)$

Multiplying/Dividing Algebraic Fractions

Grade 8

1) To multiply two fractions, just multiply tops and bottoms separately.
2) To divide, turn the second fraction upside down then multiply.

EXAMPLE: Simplify $\frac{x^2-4}{x^2+x-12} \div \frac{2x+4}{x^2-3x}$

Turn the second fraction upside down — Factorise and cancel — Multiply tops and bottoms

$$\frac{x^2-4}{x^2+x-12} \div \frac{2x+4}{x^2-3x} = \frac{x^2-4}{x^2+x-12} \times \frac{x^2-3x}{2x+4} = \frac{(x+2)(x-2)}{(x+4)(x-3)} \times \frac{x(x-3)}{2(x+2)} = \frac{x-2}{x+4} \times \frac{x}{2} = \frac{x(x-2)}{2(x+4)}$$

Adding/Subtracting Algebraic Fractions

Grade 8

For the common denominator, find something both denominators divide into.

Adding or subtracting is a bit more difficult:

1) Work out the common denominator (see p.6).
2) Multiply top and bottom of each fraction by whatever gives you the common denominator.
3) Add or subtract the numerators only.

Fractions		
$\frac{1}{x} + \frac{1}{3x}$	$\frac{1}{x+1} + \frac{1}{x-2}$	$\frac{1}{x} + \frac{1}{x(x+1)}$
3x	(x + 1)(x – 2)	x(x + 1)
Common denominator		

EXAMPLE: Write $\frac{3}{(x+3)} + \frac{1}{(x-2)}$ as a single fraction.

1st fraction: × top & bottom by $(x - 2)$
2nd fraction: × top & bottom by $(x + 3)$

$$\frac{3}{(x+3)} + \frac{1}{(x-2)} = \frac{3(x-2)}{(x+3)(x-2)} + \frac{(x+3)}{(x+3)(x-2)}$$

Common denominator will be $(x + 3)(x - 2)$

Add the numerators

$$= \frac{3x-6}{(x+3)(x-2)} + \frac{x+3}{(x+3)(x-2)} = \frac{4x-3}{(x+3)(x-2)}$$

Factorising Quadratics

4

When 'a' is Not 1

Grade 7

The basic method is still the same but it's a bit messier — the initial brackets are different as the first terms in each bracket have to multiply to give 'a'. This means finding the other numbers to go in the brackets is harder as there are more combinations to try. The best way to get to grips with it is to have a look at an example.

EXAMPLE: Solve $3x^2 + 7x - 6 = 0$.

1) $3x^2 + 7x - 6 = 0$
2) $(3x \quad)(x \quad) = 0$
3) Number pairs: 1 × 6 and 2 × 3

(3x 1)(x 6) multiplies to give 18x and 1x
which add/subtract to give 17x or 19x
(3x 6)(x 1) multiplies to give 3x and 6x
which add/subtract to give 9x or 3x
(3x 3)(x 2) multiplies to give 6x and 3x
which add/subtract to give 9x or 3x
(3x 2)(x 3) multiplies to give 9x and 2x
which add/subtract to give 11x or 7x ✓

(3x 2)(x 3)

4) $(3x - 2)(x + 3)$
5) $(3x - 2)(x + 3) = 3x^2 + 9x - 2x - 6$
$= 3x^2 + 7x - 6$ ✓
6) $(3x - 2) = 0 \Rightarrow x = \frac{2}{3}$
$(x + 3) = 0 \Rightarrow x = -3$

1) Rearrange into the standard format.
2) Write down the initial brackets — this time, one of the brackets will have a 3x in it.
3) The tricky part: first, find pairs of numbers that multiply to give c (= 6), ignoring the minus sign for now.
Then, try out the number pairs you just found in the brackets until you find one that gives 7x. But remember, each pair of numbers has to be tried in 2 positions (as the brackets are different — one has 3x in it).
4) Now fill in the +/– signs so that 9 and 2 add/subtract to give +7 (= b).
5) ESSENTIAL check — EXPAND the brackets.
6) SOLVE THE EQUATION by setting each bracket equal to 0 (if a isn't 1, one of your answers will be a fraction).

Quadratic Formula — Five Crucial Details

Grade 7

1) Take it nice and slowly — always write it down in stages as you go.
2) **WHENEVER YOU GET A MINUS SIGN, THE ALARM BELLS SHOULD ALWAYS RING!**
3) Remember it's '2a' on the bottom line, not just 'a' — and you divide ALL of the top line by 2a.
4) The ± sign means you end up with two solutions (by replacing it in the final step with '+' and '–').
5) If you get a negative number inside your square root, go back and check your working. Some quadratics do have a negative value in the square root, but they won't come up at GCSE.

If either 'a' or 'c' is negative, the -4ac effectively becomes +4ac, so watch out. Also, be careful if b is negative, as -b will be positive.

5

EXAMPLE: Solve $3x^2 + 7x = 1$, giving your answers to 2 decimal places.

$3x^2 + 7x - 1 = 0$

$a = 3, \quad b = 7, \quad c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-7 \pm \sqrt{7^2 - 4 \times 3 \times -1}}{2 \times 3}$$
$$= \frac{-7 \pm \sqrt{49 + 12}}{6}$$
$$= \frac{-7 \pm \sqrt{61}}{6}$$
$$= \frac{-7 + \sqrt{61}}{6} \text{ or } \frac{-7 - \sqrt{61}}{6}$$
$$= 0.1350\ldots \text{ or } -2.468\ldots$$

So to 2 d.p. the solutions are: $x = 0.14$ or -2.47

Notice that you do two calculations at the final stage — one + and one –.

1) First get it into the form $ax^2 + bx + c = 0$.
2) Then carefully identify a, b and c.
3) Put these values into the quadratic formula and write down each stage.
4) Finally, as a check put these values back into the original equation:
E.g. for x = 0.1350: $3 \times 0.135^2 + 7 \times 0.135 = 0.999675$, which is 1, as near as...

When to use the quadratic formula:

- If you have a quadratic that won't easily factorise.
- If the question mentions decimal places or significant figures.
- If the question asks for exact answers or surds (though this could be completing the square instead — see next page).

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Finding the nth Term of a Quadratic Sequence (Grade 7)

6

A quadratic sequence has an n^2 term — the difference between the terms changes as you go through the sequence, but the difference between the differences is the same each time.

EXAMPLE: Find an expression for the nth term of the sequence that starts 10, 14, 20, 28...

n: 1 2 3 4

term: 10 14 20 28

differences: +4 +6 +8

second differences: +2 +2 — So the expression will contain an n^2 term.

term:	10	14	20	28
n^2:	1	4	9	16
term – n^2:	9	10	11	12

The expression for this linear sequence is n + 8

So the expression for the nth term is $n^2 + n + 8$

1) Find the difference between each pair of terms.
2) The difference is changing, so work out the difference between the differences.
3) Divide this value by 2 — this gives the coefficient of the n^2 term (here it's 2 ÷ 2 = 1).
4) Subtract the n^2 term from each term in the sequence. This will give you a linear sequence.
5) Find the rule for the nth term of the linear sequence (see above) and add this on to the n^2 term.

Again, make sure you check your expression by putting the first few values of n back in — so n = 1 gives $1^2 + 1 + 8 = 10$, n = 2 gives $2^2 + 2 + 8 = 14$ and so on.

Show Things Are Odd, Even or Multiples by Rearranging

7

Before you get started, there are a few things you need to know — they'll come in very handy when you're trying to prove things.

- Any even number can be written as 2n — i.e. 2 × something.
- Any odd number can be written as 2n + 1 — i.e. 2 × something + 1.
- Consecutive numbers can be written as n, n + 1, n + 2 etc. — you can apply this to e.g. consecutive even numbers too (they'd be written as 2n, 2n + 2, 2n + 4). (In all of these statements, n is just any integer.)
- The sum, difference and product of integers is always an integer.

This can be extended to multiples of other numbers too — e.g. to prove that something is a multiple of 5, show that it can be written as 5 × something.

EXAMPLE: (Grade 6) Prove that the sum of any three odd numbers is odd.

So what you're trying to do here is show that the sum of three odd numbers can be written as (2 × integer) + 1.

Take three odd numbers:
2a + 1, 2b + 1 and 2c + 1
(they don't have to be consecutive)

Add them together:
$2a + 1 + 2b + 1 + 2c + 1 = 2a + 2b + 2c + 2 + 1$
$= 2(a + b + c + 1) + 1$
$= 2n + 1$ where n is an integer $(a + b + c + 1)$

You'll see why I've written 3 as 2 + 1 in a second.

So the sum of any three odd numbers is odd.

EXAMPLE: (Grade 5) Prove that $(n + 3)^2 - (n - 2)^2 \equiv 5(2n + 1)$.

Take one side of the equation and play about with it until you get the other side:

LHS: $(n + 3)^2 - (n - 2)^2 \equiv n^2 + 6n + 9 - (n^2 - 4n + 4)$
$\equiv n^2 + 6n + 9 - n^2 + 4n - 4$
$\equiv 10n + 5$
$\equiv 5(2n + 1)$ = RHS ✓

≡ is the identity symbol, and means that two things are identically equal to each other. So a + b ≡ b + a is true for all values of a and b (unlike an equation, which is only true for certain values).

Direct Proportion (Grade 4)

8

1) Two quantities, A and B, are in direct proportion (or just in proportion) if increasing one increases the other one proportionally. So if quantity A is doubled (or trebled, halved, etc.), so is quantity B.
2) Remember this golden rule for direct proportion questions:

DIVIDE for ONE, then TIMES for ALL

EXAMPLE: Hannah pays £3.60 per 400 g of cheese. She uses 220 g of cheese to make 4 cheese pasties. How much would the cheese cost if she wanted to make 50 cheese pasties?

In 1 pasty there is:	220 g ÷ 4 = 55 g of cheese
So in 50 pasties there is:	55 g × 50 = 2750 g of cheese
1 g of cheese would cost:	£3.60 ÷ 400 = 0.9p
So 2750 g of cheese would cost:	0.9 × 2750 = 2475p = £24.75

There will often be lots of stages to direct proportion questions — keep track of what you've worked out at each stage.

Inverse Proportion (Grade 4)

9

1) Two quantities, C and D, are in inverse proportion if increasing one quantity causes the other quantity to decrease proportionally. So if quantity C is doubled (or tripled, halved, etc.), quantity D is halved (or divided by 3, doubled etc.).
2) The rule for finding inverse proportions is:

TIMES for ONE, then DIVIDE for ALL

EXAMPLE: 4 bakers can decorate 100 cakes in 5 hours.

a) How long would it take 10 bakers to decorate the same number of cakes?

100 cakes will take 1 baker:	5 × 4 = 20 hours
So 100 cakes will take 10 bakers:	20 ÷ 10 = 2 hours for 10 bakers

b) How long would it take 11 bakers to decorate 220 cakes?

100 cakes will take 1 baker:	20 hours
1 cake will take 1 baker:	20 ÷ 100 = 0.2 hours
220 cakes will take 1 baker:	0.2 × 220 = 44 hours
220 cakes will take 11 bakers:	44 ÷ 11 = 4 hours

The number of bakers is inversely proportional to number of hours — but the number of cakes is directly proportional to the number of hours.

Completing the Square

There's just one more method to learn for solving quadratics — and it's a bit of a nasty one. It's called 'completing the square', and takes a bit to get your head round it.

Solving Quadratics by 'Completing the Square' (Grade 8)

To 'complete the square' you have to:

1) Write down a SQUARED bracket, and then 2) Stick a number on the end to 'COMPLETE' it.

$$x^2 + 12x - 5 = (x + 6)^2 - 41$$

The SQUARE... ...COMPLETED

It's not that bad if you learn all the steps — some of them aren't all that obvious.

1) As always, **REARRANGE THE QUADRATIC INTO THE STANDARD FORMAT**: $ax^2 + bx + c$ (the rest of this method is for $a = 1$).
2) **WRITE OUT THE INITIAL BRACKET**: $(x + \frac{b}{2})^2$ — just divide the value of b by 2.
3) **MULTIPLY OUT THE BRACKETS** and **COMPARE TO THE ORIGINAL** to find what you need to add or subtract to complete the square.
4) Add or subtract the **ADJUSTING NUMBER** to make it **MATCH THE ORIGINAL**.

If a isn't 1, you have to divide through by 'a' or take out a factor of 'a' at the start — see next page.

EXAMPLE: a) Express $x^2 + 8x + 5$ in the form $(x + m)^2 + n$.

1) It's in the standard format. — $x^2 + 8x + 5$
2) Write out the initial bracket — $(x + 4)^2$
3) Multiply out the brackets and compare to the original. — $(x + 4)^2 = x^2 + 8x + 16$ (Original equation had +5 here...)
4) Subtract adjusting number (11). — $(x + 4)^2 - 11 = x^2 + 8x + 16 - 11$ (...so you need -11)
$= x^2 + 8x + 5$ ✓ — matches original now!

So the completed square is: $(x + 4)^2 - 11$.

b) Hence solve $x^2 + 8x + 5 = 0$, leaving your answers in surd form.

Now use the completed square to solve the equation. There are three more steps for this:

$(x + 4)^2 - 11 = 0$

1) Put the number on the other side (+11). — $(x + 4)^2 = 11$
2) Square root both sides (don't forget the ±!) ($\sqrt{\ }$). — $x + 4 = \pm\sqrt{11}$
3) Get x on its own (–4). — $x = -4 \pm \sqrt{11}$

So the two solutions (in surd form) are: $x = -4 + \sqrt{11}$ and $x = -4 - \sqrt{11}$

Completing the Square When 'a' Isn't 1 (Grade 9)

If 'a' isn't 1, completing the square is a bit trickier. You follow the same method as on the previous page, but you have to take out a factor of 'a' from the x^2 and x-terms before you start (which often means you end up with awkward fractions). This time, the number in the brackets is $\frac{b}{2a}$.

EXAMPLE: Write $2x^2 + 5x + 9$ in the form $a(x + m)^2 + n$.

1) It's in the standard format. — $2x^2 + 5x + 9$
2) Take out a factor of 2. — $2(x^2 + \frac{5}{2}x) + 9$
3) Write out the initial bracket. — $2(x + \frac{5}{4})^2$
4) Multiply out the bracket and compare to the original. — $2(x + \frac{5}{4})^2 = 2x^2 + 5x + \frac{25}{8}$ (Original equation had +9 here...)
5) Add on adjusting number ($\frac{47}{8}$). — $2(x + \frac{5}{4})^2 + \frac{47}{8} = 2x^2 + 5x + \frac{25}{8} + \frac{47}{8}$ (...so you need $9 - \frac{25}{8} = \frac{47}{8}$)
$= 2x^2 + 5x + 9$ ✓ — matches original now!

So the completed square is: $2(x + \frac{5}{4})^2 + \frac{47}{8}$

The Completed Square Helps You Sketch the Graph (Grade 9)

There's more about sketching quadratic graphs on p.48, but you can use the completed square to work out key details about the graph — like the turning point (maximum or minimum) and whether it crosses the x-axis.

1) For a positive quadratic (where the x^2 coefficient is positive), the adjusting number tells you the minimum y-value of the graph. If the completed square is $a(x + m)^2 + n$, this minimum y-value will occur when the brackets are equal to 0 (because the bit in brackets is squared, so is never negative) — i.e. when $x = -m$.
2) The solutions to the equation tell you where the graph crosses the x-axis. If the adjusting number is positive, the graph will never cross the x-axis as it will always be greater than 0 (this means that the quadratic has no real roots).

EXAMPLE: Sketch the graph of $y = 2x^2 + 5x + 9$.

This is only a sketch, so label the points you know

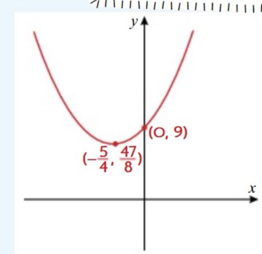
From above, completed square form is $2(x + \frac{5}{4})^2 + \frac{47}{8}$.

The minimum point occurs when the brackets are equal to 0 — this will happen when $x = -\frac{5}{4}$.

At this point, the graph takes its minimum value, which is the adjusting number ($\frac{47}{8}$).

The adjusting number is positive, so the graph will never cross the x-axis.

Find where the curve crosses the y-axis by substituting $x = 0$ into the equation and mark this on your graph. $y = 0 + 0 + 9 = 9$



How do we use Knowledge Organisers in Mathematics

How can you use knowledge organisers at home to help us?

- **Retrieval Practice**: Read over a section of the knowledge organiser, cover it up and then write down everything you can remember. Repeat until you remember everything.
- **Flash Cards:** Using the Knowledge Organisers to help on one side of a piece of paper write a question, on the other side write an answer. Ask someone to test you by asking a question and seeing if you know the answer.
- **Mind Maps:** Turn the information from the knowledge organiser into a mind map. Then reread the mind map and on a piece of paper half the size try and recreate the key phrases of the mind map from memory.
- **Sketch it:** Draw an image to represent each fact; this can be done in isolation or as part of the mind map/flash card.
- **Teach it:** Teach someone the information on your knowledge organiser, let them ask you questions and see if you know the answers.

How will we use knowledge organisers in Mathematics?

Knowledge organisers will be used before I complete a Learning Check or Common Assessment. I will spend part of the lesson looking over each of the key topics of the half term before completing the Learning Check or Common Assessment.

I will also use these at home to complete my own independent learning and revision of these key topics.

Year 10 Mathematics (Higher): Low Stake Test scores (Autumn)

Topics	Date	Score
The Sine Rule, The Cosine Rule, Area of a Triangle ($A = \frac{1}{2}abSinC$), Direct Proportion and Inverse Proportion.		
Factorising Harder Quadratics, The Quadratic Formula, Completing the Square, Algebraic Proof and Quadratic nth Term.		
Simplifying Algebraic Fractions, Multiplying Algebraic Fractions, Dividing Algebraic Fractions, Adding Algebraic Fractions and Subtracting Algebraic Fractions		

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$