

Knowledge Organiser: Year 10 (Higher) - Half Term 1



The square of a number

To square a number, we just multiply it by itself.

For example, to square 5 we multiply 5 by 5. We get an answer of 25.

We can use mathematical notation to save time when squaring numbers. Instead of writing the square of 5 – or 5 squared – we can instead write 5^2 .

Example 1

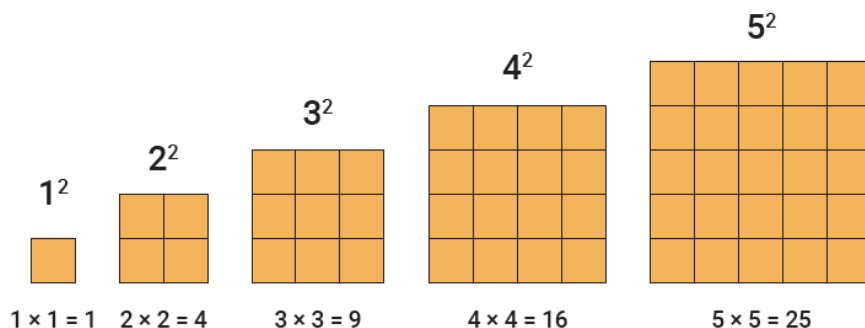
Find the value of the square of 8.
 $8^2 = 8 \times 8 = 64$

Example 2

What is the value of 3^2 ?
 $3^2 = 3 \times 3 = 9$

The answer we get when we square a whole number is called a square number. The first 5 square numbers are 1, 4, 9, 16 and 25.

Square numbers are ones that you can arrange into a square.



The square root of a number

Finding the square root of a number is the **inverse** of squaring a number.

For example, 5 squared is 25. The square root of 25 is 5.
6 squared is 36. The square root of 36 is 6.

We can also use the square root symbol to save time.
For example, instead of writing the square root of 36, we write $\sqrt{36}$.

The cube of a number

A cube number is a number multiplied by itself three times. This can also be called a number cubed.

For example, to calculate the value of 2 cubed we multiply 2 by 2 by 2. We get an answer of 8.

We can also use mathematical notation to save time.

For example, instead of writing the cube of 5, or 5 cubed, we can write 5^3 .

Example 1

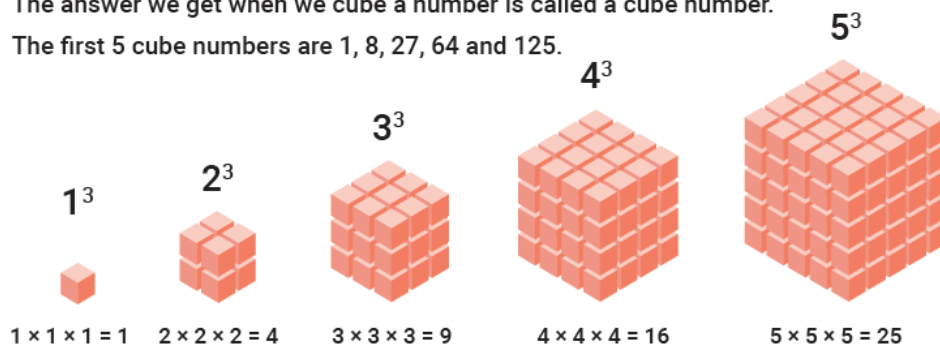
Find the value of the cube of 3.
 $3^3 = 3 \times 3 \times 3 = 27$

Example 2

Find the value of 4^3 .
 $4^3 = 4 \times 4 \times 4 = 64$.

The answer we get when we cube a number is called a cube number.

The first 5 cube numbers are 1, 8, 27, 64 and 125.



The cube root of a number

Finding the cube root of a number is the inverse of cubing a number.

For example, the cube root of 125 is 5 because 5 cubed is 125.
The cube root of 8 is 2 because 2 cubed is 8.

We can also use the cube root symbol to save time.
For example, instead of writing the cube root of 8, we can write $\sqrt[3]{8}$.

Multiplying numbers with powers

If we consider $4^3 \times 4^2$ then this gives $(4 \times 4 \times 4) \times (4 \times 4) = 4^5$

Therefore, when we multiply the same base number the powers are added.

$$a^m \times a^n = a^{m+n}$$

E.g., simplify

$$1) 3^4 \times 3^3 = 3^{3+4} = 3^7$$

$$2) 2^{-2} \times 2^6 = 2^{-2+6} = 2^4$$

Dividing numbers with powers

If we consider $7^5 \div 7^2$ then this gives

$$\frac{\cancel{7} \times \cancel{7} \times 7 \times 7 \times 7}{\cancel{7} \times \cancel{7}} = 7^3$$

$$* \text{ Remember } \frac{7}{7} = 1$$

Therefore, when we divide the same base number the powers are subtracted.

$$a^m \div a^n = a^{m-n}$$

E.g., simplify

$$1) 8^7 \div 8^5 = 8^{7-5} = 8^2$$

$$2) 5^3 \div 5^{-6} = 5^{3-(-6)} = 5^9$$

Raising a power to a power

If we consider $(6^3)^5$ then this gives

$$6^3 \times 6^3 \times 6^3 \times 6^3 \times 6^3 = 6^{3+3+3+3+3} \text{ or } 6^{3 \times 5} = 6^{15}$$

Therefore, when we raise a power to another power the powers are multiplied.

$$(a^m)^n = a^{mn}$$

E.g., simplify

$$1) (4^5)^2 = 4^{5 \times 2} = 4^{10}$$

$$2) (9^3)^4 = 9^{3 \times 4} = 9^{12}$$

Powers that are 0

If we consider $5^4 \div 5^4$ then rules of indices tell us that

$$5^4 \div 5^4 = 5^{4-4} = 5^0.$$

If we look at the division like this

$$\frac{\cancel{5} \times \cancel{5} \times \cancel{5} \times \cancel{5}}{\cancel{5} \times \cancel{5} \times \cancel{5} \times \cancel{5}} = 1, \text{ then this shows that } 5^0 = 1.$$

Therefore, any number to the power of 0 will always equal 1 $a^0 = 1$

Using more than one rule

E.g., simplify

$$1) \frac{3^5 \times 3^6}{3^7} = \frac{3^{11}}{3^7} = 3^4$$

$$2) \frac{(8^3)^4}{8^{15} \times 8^{-3}} = \frac{8^{12}}{8^{12}} = 8^0 = 1$$

Multiplication

$$1) (5 \times 10^4) \times (3 \times 10^2) = 5 \times 3 \times 10^4 \times 10^2 = 15 \times 10^{4+2} = 15 \times 10^6 = 1.5 \times 10^7$$

$$2) (6.4 \times 10^5) \times (2 \times 10^{-3}) = 6.4 \times 2 \times 10^5 \times 10^2 = 12.8 \times 10^{5+(-3)} = 12.8 \times 10^2 = 1.28 \times 10^3$$

Division

$$3) (2 \times 10^9) \div (8 \times 10^6) = 2 \div 8 \times 10^9 \div 10^6 = 0.25 \times 10^{9-6} = 0.25 \times 10^3 = 2.5 \times 10^2$$

$$4) (9 \times 10^{-4}) \div (3 \times 10^7) = 9 \div 3 \times 10^{-4} \div 10^7 = 3 \times 10^{-4-7} = 3 \times 10^{-11}$$

Addition

$$5) (5 \times 10^4) + (6.2 \times 10^3) = 50\,000 + 6\,200 = 56\,200 = 5.62 \times 10^4$$

$$6) (3.2 \times 10^{-4}) + (5.4 \times 10^{-3}) = 0.00032 + 0.0054 = 0.00572 = 5.72 \times 10^{-3}$$

Subtraction

$$7) (7.8 \times 10^4) - (5 \times 10^2) = 78\,000 - 500 = 77\,500 = 7.75 \times 10^4$$

$$8) (2.4 \times 10^{-2}) - (1.8 \times 10^{-3}) = 0.024 - 0.0018 = 0.0222 = 2.22 \times 10^{-2}$$

Reciprocals and negative powers

Negative powers are used to represent reciprocals, i.e., $1 \div$ by the number.

The reciprocal of 2 can be written as $2^{-1} = \frac{1}{2}$
 The reciprocal of 2^2 can be written as $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
 The reciprocal of 2^3 can be written as $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

Therefore, for negative powers: $a^{-n} = \frac{1}{a^n}$

E.g., Calculate the following

$$1) 7^{-1} = \frac{1}{7} \quad 2) \frac{2^{-1}}{5} = \frac{\frac{1}{2}}{5} = \frac{1}{2 \times 5} = \frac{1}{10}$$

$$3) 3^{-3} = \frac{1}{3^3} = \frac{1}{27} \quad 4) 2^{-6} = \frac{1}{2^6} = \frac{1}{64}$$

$$5) 5^{-4} = \frac{1}{5^4} = \frac{1}{625} \quad 6) 10^{-5} = \frac{1}{10^5} = \frac{1}{100\,000}$$

Fractional powers and roots

Fractional powers are used to represent roots.

The square root of 4 can be written as $4^{\frac{1}{2}} = \sqrt{4} = 2$

The cube root of 8 can be written as $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

Therefore, for fractional powers: $\frac{1}{a^n} = a^{-n}$

Fractional powers are also used to show a root raised to a power:

$$\frac{m}{a^n} = (a^{\frac{1}{n}})^m = ({}^n\sqrt{a})^m$$

E.g., evaluate the following

$$1) 144^{\frac{1}{2}} = \sqrt{144} = 12 \quad 2) 64^{\frac{1}{3}} = \sqrt[3]{64} = 4$$

$$3) 81^{\frac{1}{4}} = \sqrt[4]{81} = 3 \quad 4) 125^{\frac{2}{3}} = (\sqrt[3]{125})^2 = 5^2 = 25$$

$$5) 32^{\frac{3}{5}} = (\sqrt[5]{32})^3 = 2^3 = 8 \quad 6) 36^{\frac{3}{2}} = (\sqrt{36})^3 = 6^3 = 216$$

Irrational numbers

These are numbers that cannot be written as a fraction

e.g. $\sqrt{2}$, $\sqrt{5}$, $\sqrt{2} + 1$, $\sqrt[3]{4}$, π , $\frac{\pi}{2}$, 3π .

The irrational numbers that involve square roots are called **surds**.

Rules to remember:

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\sqrt{a} \times \sqrt{a} = a$$

Top tip! When simplifying surds, look for factors that are square numbers.

Remember you can add and subtract surds if the number under the square root is the same.

Simplifying surds

E.g., simplify the following

$$1) \sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$$

$$2) \sqrt{32} + \sqrt{50} = \sqrt{2} \times \sqrt{16} + \sqrt{2} \times \sqrt{25} = 4\sqrt{2} + 5\sqrt{2} = 9\sqrt{2}$$

$$3) \sqrt{125} - \sqrt{45} = \sqrt{5} \times \sqrt{25} - \sqrt{9} \times \sqrt{5} = 5\sqrt{5} - 3\sqrt{5} = 2\sqrt{5}$$

$$4) \sqrt{\frac{12}{25}} = \frac{\sqrt{12}}{\sqrt{25}} = \frac{\sqrt{3 \times 4}}{5} = \frac{2\sqrt{3}}{5}$$

Expanding brackets

Example 1

Consider $3(2x + 5)$.

This means the same as $3 \times (2x + 5)$.

We have:

$$3(2x + 5) = 3 \times 2x + 3 \times 5$$

This gives a final answer of $6x + 15$.

$6x$ and 15 are different terms within the expression $6x + 15$.

Expanding brackets involving negative numbers

Care is needed if there is a negative number either inside or outside the brackets.

Example 2

Expand $4(3x - 5)$.

Answer:

$$4(3x - 5) = 4 \times 3x + 4 \times (-5) = 12x - 20$$

Example 3

Expand $-2(x - 6)$.

Answer:

$$-2(x - 6) = (-2) \times x + (-2) \times (-6) = -2x + 12$$

This could also be written as $12 - 2x$.

Example 4

Expand $-(7x - 3)$.

Answer:

Remember that

$-(7x - 3)$ is the same as $-1(7x - 3)$.

$$-(7x - 3) = (-1) \times 7x + (-1) \times (-3) = -7x + 3$$

Factors

Numbers that divide exactly into another number with no remainder.

E.g., the factors of 18 are...

1, 2, 3, 6, 9 and 18

Highest common factor (HCF)

The highest number to appear in both lists of factors.

E.g., the HCF of 18 and 45 is 9.

Factors of 18



1, 2, 3, 6, 9, 18

Factors of 45



1, 3, 5, 9, 15, 45

Prime numbers

Prime numbers have only 2 factors, 1 and the number itself.

E.g., **2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 ...**

Factorising by finding the highest common factor

When every term in an expression has a common factor it is possible to factorise the expression by placing the highest common factor outside of the brackets. This is the largest number or letter (or both) that divides exactly into every term of the expression. There are many methods to factorise, only one of which is given here.

1

Factorise $6x + 18$

1, 2, 3 and 6 are all numbers that divide into both terms, but **6** is the highest common factor therefore this is placed outside of the brackets.

$$6(\quad)$$

To complete the brackets, we must find the terms that multiply with **6** to give $6x$ ($6 \times x$) and to give $+18$ (6×3)

$$\text{Therefore } 6x + 18 = 6(x + 3)$$

2

Factorise $8x^2 - 12x$

1, 2 and 4 are all numbers that divide into both terms in the expression, with 4 being the largest. Both terms are also divisible by x , therefore **$4x$** is the highest common factor of $8x^2$ and $-12x$. This is the term placed outside of the brackets.

$$4x(\quad)$$

To complete the brackets, we must find the terms that multiply with **$4x$** to give $8x^2$ ($4x \times 2x$) and $-12x$ ($4x \times -3$)

$$\text{Therefore } 8x^2 - 12x = 4x(2x - 3)$$

3

Factorise $6a^2b + 15ac^2$

1 and 3 are the numbers that divide into both terms in the expression, with 3 being the largest. Both terms are also divisible by a , therefore **$3a$** is the highest common factor and this is the term placed outside of the brackets.

$$3a(\quad)$$

To complete the brackets we must find the terms that multiply with **$3a$** to give $6a^2b$ ($3a \times 2ab$) and $+15ac^2$ ($3a \times 5c^2$)

$$\text{Therefore } 6a^2b + 15ac^2 = 3a(2ab + 5c^2)$$

Double brackets are trickier than single brackets — this time, you have to multiply **everything** in the **first bracket** by **everything** in the **second bracket**. You'll get **4 terms**, and usually 2 of them will combine to leave **3 terms**. There's a handy way to multiply out double brackets — it's called the **FOIL method**:

First — multiply the first term in each bracket together

Outside — multiply the outside terms (i.e. the first term in the first bracket by the second term in the second bracket)

Inside — multiply the inside terms (i.e. the second term in the first bracket by the first term in the second bracket)

Last — multiply the second term in each bracket together

EXAMPLE

Expand and simplify $(2p - 4)(3p + 1)$

$$\begin{aligned} (2p - 4)(3p + 1) &= (2p \times 3p) + (2p \times 1) + (-4 \times 3p) + (-4 \times 1) \\ &= 6p^2 + 2p - 12p - 4 \\ &= 6p^2 - 10p - 4 \end{aligned}$$

The two p terms combine together.



Uses of foil: expanding brackets, jacket potatoes...

Always write out **SQUARED BRACKETS** as **TWO BRACKETS** (to avoid mistakes), then multiply out as above.

So $(3x + 5)^2 = (3x + 5)(3x + 5) = 9x^2 + 15x + 15x + 25 = 9x^2 + 30x + 25$.

(DON'T make the mistake of thinking that $(3x + 5)^2 = 9x^2 + 25$ — this is **wrong wrong wrong**.)

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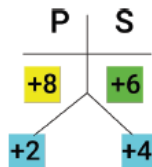
Factorising quadratic expressions

A quadratic expression can take the form $x^2 + ax + b$. There are many possible methods to factorise a quadratic expression, only one of which is given here.

1

Factorise $x^2 + 6x + 8$.

Here you need two numbers with a product of **+8** and a sum of **+6**.



(If you can't find the numbers immediately, try listing factor pairs of +8, namely 1×8 , -1×-8 , 2×4 , -2×-4)

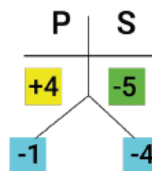
Place the two numbers in brackets like this

$$(x + 2)(x + 4) \quad \text{or} \quad (x + 4)(x + 2)$$

2

Factorise $x^2 - 5x + 4$

Here you need two numbers with a product of **+4** and a sum of **-5**.



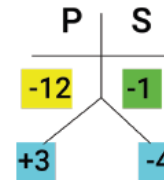
Place the two numbers in brackets like this

$$(x - 1)(x - 4) \quad \text{or} \quad (x - 4)(x - 1)$$

3

Factorise $x^2 - x - 12$

Here you need two numbers with a product of **-12** and a sum of **-1**.



Place the two numbers in brackets like this

$$(x - 4)(x + 3) \quad \text{or} \quad (x + 3)(x - 4)$$

The 'difference of two squares' (D.O.T.S. for short) is where you have 'one thing squared' take away 'another thing squared'. There's a quick and easy way to factorise it — just use the rule below:

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE

Factorise: a) $9p^2 - 16q^2$

Answer: $9p^2 - 16q^2 = (3p + 4q)(3p - 4q)$

Here you had to spot that 9 and 16 are square numbers.

b) $3x^2 - 75y^2$

Answer: $3x^2 - 75y^2 = 3(x^2 - 25y^2) = 3(x + 5y)(x - 5y)$

This time, you had to take out a factor of 3 first.

c) $x^2 - 5$

Answer: $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$

Although 5 isn't a square number, you can write it as $(\sqrt{5})^2$.

Watch out — the difference of two squares can creep into other algebra questions. A popular exam question is to put a difference of two squares on the top or bottom of a fraction and ask you to simplify it. There's more on algebraic fractions on p.30.

EXAMPLE

Simplify $\frac{x^2 - 36}{5x + 30}$

The numerator is a difference of two squares.

$$\frac{x^2 - 36}{5x + 30} = \frac{(x + 6)(x - 6)}{5(x + 6)} = \frac{x - 6}{5}$$

Factorise the denominator.

- 1) 'Factorising a quadratic' means 'putting it into 2 brackets'.
- 2) The standard format for quadratic equations is: $ax^2 + bx + c = 0$. E.g. $x^2 + 3x + 2 = 0$
- 3) If $a = 1$, the quadratic is much easier to deal with. You might be asked to solve the equation. This just means finding the values of x that make each bracket 0 (see example below).
- 4) As well as factorising a quadratic, you might be asked to solve the equation.

The basic method is still the same but it's a bit messier — the initial brackets are different as the first terms in each bracket have to multiply to give 'a'. This means finding the other numbers to go in the brackets is harder as there are more combinations to try. The best way to get to grips with it is to have a look at an example.

EXAMPLE Solve $3x^2 + 7x - 6 = 0$.

- 1) $3x^2 + 7x - 6 = 0$
 - 2) $(3x \quad)(x \quad) = 0$
 - 3) Number pairs: 1×6 and 2×3
 - 4) $(3x - 2)(x + 3)$
 - 5) $(3x - 2)(x + 3) = 3x^2 + 9x - 2x - 6 = 3x^2 + 7x - 6$
 - 6) $(3x - 2) = 0 \Rightarrow x = \frac{2}{3}$
 $(x + 3) = 0 \Rightarrow x = -3$
- 1) Rearrange into the standard format.
- 2) Write down the initial brackets — this time, one of the brackets will have a 3x in it.
- 3) The tricky part: first, find pairs of numbers that multiply to give c ($= 6$), ignoring the minus sign for now.
- Then, try out the number pairs you just found in the brackets until you find one that gives $7x$. But remember, each pair of numbers has to be tried in 2 positions (as the brackets are different — one has $3x$ in it).
- 4) Now fill in the +/− signs so that 9 and 2 add/subtract to give $+7$ ($= b$).
- 5) ESSENTIAL check — EXPAND the brackets.
- 6) SOLVE THE EQUATION by setting each bracket equal to 0 (if a isn't 1, one of your answers will be a fraction).

EXAMPLE Solve $2x^2 - 9x = 5$.

- 1) Put in standard form: $2x^2 - 9x - 5 = 0$
 - 2) Initial brackets: $(2x \quad)(x \quad) = 0$
 - 3) Number pairs: 1×5
 - 4) Put in the signs: $(2x + 1)(x - 5)$
 - 5) Check: $(2x + 1)(x - 5) = 2x^2 - 10x + x - 5 = 2x^2 - 9x - 5$
 - 6) Solve: $(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}$
 $(x - 5) = 0 \Rightarrow x = 5$
- $(2x \quad 5)(x \quad 1)$ multiplies to give $2x$ and $5x$ which add/subtract to give $3x$ or $7x$
- $(2x \quad 1)(x \quad 5)$ multiplies to give $1x$ and $10x$ which add/subtract to give $9x$ or $11x$
- $(2x \quad 1)(x \quad 5)$ ✓

There's more about sketching quadratic graphs on p.48, but you can use the completed square to work out key details about the graph — like the turning point (maximum or minimum) and whether it crosses the x-axis.

- 1) For a positive quadratic (where the x^2 coefficient is positive), the adjusting number tells you the minimum y-value of the graph. If the completed square is $a(x + m)^2 + n$, this minimum y-value will occur when the brackets are equal to 0 (because the bit in brackets is squared, so is never negative) — i.e. when $x = -m$.
- 2) The solutions to the equation tell you where the graph crosses the x-axis. If the adjusting number is positive, the graph will never cross the x-axis as it will always be greater than 0 (this means that the quadratic has no real roots).

To 'complete the square' you have to:

- 1) Write down a SQUARED bracket, and then...
- 2) Stick a number on the end to 'COMPLETE' it.

$$x^2 + 12x - 5 = (x + 6)^2 - 41$$

The SQUARE... ...COMPLETED

It's not that bad if you learn all the steps — some of them aren't all that obvious.

- 1) As always, REARRANGE THE QUADRATIC INTO THE STANDARD FORMAT: $ax^2 + bx + c$ (the rest of this method is for $a = 1$).
- 2) WRITE OUT THE INITIAL BRACKET: $(x + \frac{b}{2})^2$ — just divide the value of b by 2.
- 3) MULTIPLY OUT THE BRACKETS and COMPARE TO THE ORIGINAL to find what you need to add or subtract to complete the square.
- 4) Add or subtract the ADJUSTING NUMBER to make it MATCH THE ORIGINAL.

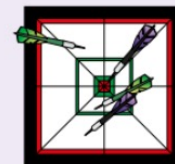
If a isn't 1, you have to divide through by 'a' or take out a factor of 'a' at the start — see next page.

EXAMPLE a) Express $x^2 + 8x + 5$ in the form $(x + m)^2 + n$.

- 1) It's in the standard format.
 - 2) Write out the initial bracket.
 - 3) Multiply out the brackets and compare to the original.
 - 4) Subtract adjusting number (II).
- $x^2 + 8x + 5$
- $(x + 4)^2$
- $(x + 4)^2 = x^2 + 8x + 16$
- $(x + 4)^2 - 11 = x^2 + 8x + 16 - 11$...so you need -11
- $= x^2 + 8x + 5$ ✓ matches original now!
- So the completed square is: $(x + 4)^2 - 11$.

Now use the completed square to solve the equation. There are three more steps for this:

- 1) Put the number on the other side (+11).
 - 2) Square root both sides (don't forget the $\pm$) ($\sqrt{\quad}$).
 - 3) Get x on its own (-4).
- $(x + 4)^2 - 11 = 0$
- $(x + 4)^2 = 11$
- $x + 4 = \pm\sqrt{11}$
- $x = -4 \pm \sqrt{11}$
- So the two solutions (in surd form) are: $x = -4 + \sqrt{11}$ and $x = -4 - \sqrt{11}$



EXAMPLE Sketch the graph of $y = 2x^2 + 5x + 9$.

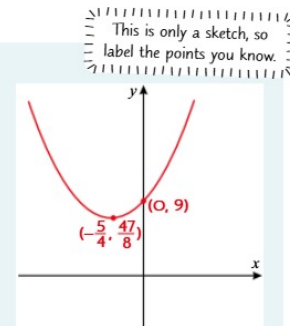
From above, completed square form is $2(x + \frac{5}{4})^2 + \frac{47}{8}$.

The minimum point occurs when the brackets are equal to 0 — this will happen when $x = -\frac{5}{4}$.

At this point, the graph takes its minimum value, which is the adjusting number ($\frac{47}{8}$).

The adjusting number is positive, so the graph will never cross the x-axis.

Find where the curve crosses the y-axis by substituting $x = 0$ into the equation and mark this on your graph. $y = 0 + 0 + 9 = 9$





How do we use Knowledge Organisers in Mathematics

How can you use knowledge organisers at home to help us?

- **Retrieval Practice:** Read over a section of the knowledge organiser, cover it up and then write down everything you can remember. Repeat until you remember everything.
- **Flash Cards:** Using the Knowledge Organisers to help on one side of a piece of paper write a question, on the other side write an answer. Ask someone to test you by asking a question and seeing if you know the answer.
- **Mind Maps:** Turn the information from the knowledge organiser into a mind map. Then reread the mind map and on a piece of paper half the size try and recreate the key phrases of the mind map from memory.
- **Sketch it:** Draw an image to represent each fact; this can be done in isolation or as part of the mind map/flash card.
- **Teach it:** Teach someone the information on your knowledge organiser, let them ask you questions and see if you know the answers.

How will we use knowledge organisers in Mathematics?

Knowledge organisers will be used before I complete a Learning Check or Common Assessment. I will spend part of the lesson looking over each of the key topics of the half term before completing the Learning Check or Common Assessment.

I will also use these at home to complete my own independent learning and revision of these key topics.